Fractons in effective field theories for spontaneously broken translations

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Based on:

- R. Argurio, C. Hoyos, D. Musso, D. N. *Phys. Rev. D* **104** no. 10, (2021) 105001, arXiv:2107.03073 [hep-th].

Structure of the talk

1. Goldstone Physics
2. Fractons in field theory
3. Fractons in EFTs for spontaneously broken translations
4. Conclusion: results and outlooks

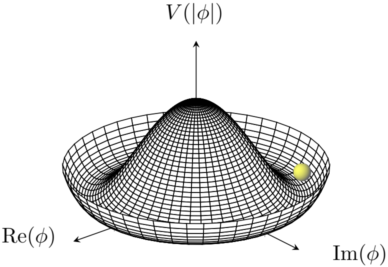
Key concepts

- We consider field theories defined on Minkowski spacetime at $T = 0$ & $\mu = 0$, with $d \geq 2$. We chose

$$\text{diag}(\eta) = (+, -, \ldots, -) \quad ; \quad \hbar = 1 = c \; .$$

- Symmetry: the EOM are unchanged by the considered transformation $\Leftarrow \delta S = 0$.

- Spontaneous symmetry breaking (SSB): a stable state of the system transforms under certain symmetries of the theory.

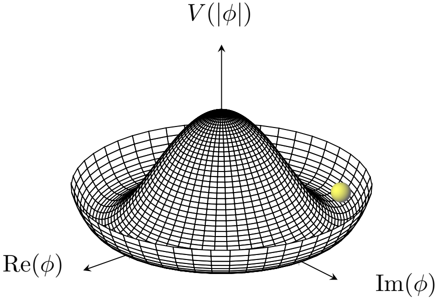


Goldstone's theorem

Goldstone's theorem [Goldstone, Nuovo Cim. '61]

If a theory has a global continuous symmetry group G spontaneously broken to a subgroup H. Then, the spectrum of the theory will contain at least one gapless mode ($\omega \xrightarrow[\vec{k}\to\vec{0}]{} 0$): Nambu Goldstone mode (NG mode).

- Let v be the VEV and X be a broken generator.
  NG candidate: $e^{i\pi(x)X}\, v$
- NG candidates are weakly coupled in the IR.



More on the NG modes

Perturbation theory around the VEV:

- NG candidates weakly coupled in the IR $\Rightarrow$ $\boxed{\mathcal{L}_{\text{int}} \sim \ldots \partial\pi \ldots}$.

- Spontaneous breaking: symmetry still there but non-linearly realized.
  Ex. for $U(1)-$SSB:

$$(v+\rho(x))e^{i\pi(x)} \xrightarrow[U(1)]{} (v+\rho(x))e^{i(\pi(x)+\alpha)}$$

$$\Rightarrow \pi(x) \xrightarrow[U(1)]{} \pi(x)+\alpha \; .$$

  NG candidates transform "shift like"

- Parametrization of the NG candidates ($\sim e^{i\pi(x)X}\, v$)
  $\neq$ parametrization diagonalizing the kinetic matrix

  $\rightarrow$ involved identification of the modes.

  $\rightarrow$ Ward-Takahashi Identities can help.

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Motivations and Research Directions

Motivations:

- Symmetries & SSB $\Rightarrow$ universal exact results.
- **Hastily said**, Goldstone physics universally describes IR physics.

$\Rightarrow$ Light mesons physics, phase transitions, stellar superfluids ...

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Research Directions:

- Counting rules.
- Conditions to have SSB (ex: Coleman's theorem).
- How do NG modes interact with non-symmetry originated particles ?

Counting rule for internal symmetries

- $\exists$ complete counting rule for the breaking of compact internal symmetries. [Watanabe et al., PRL. '12]
- Has been obtained through the most general EFT for $G \to H$:

$$\mathcal{L}_{\text{eff}}(\pi) = c_a(\pi)\partial_t\pi^a + \frac{1}{2}g_{ab}(\pi)\partial_t\pi^a\partial_t\pi^b - \frac{1}{2}\bar{g}_{ab}(\pi)\partial_i\pi^a\partial_i\pi^b + \mathcal{O}(\partial^n) \ .$$

- The reduction of the number of NG modes is purely dynamical & is due to $c_a(\pi)\partial_t\pi^a$:

$$\mathcal{L} \sim \pi^1\partial_t\pi^2 - \pi^2\partial_t\pi^1 + \dots \ .$$

$$P_a \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\pi^a)} \rightsquigarrow \text{ canonically conjugated pairs } (\pi^1, -\pi^2)\,\&\,(\pi^2, \pi^1) \ .$$

- For relativistic theories: $c_a(\pi) = 0 \Rightarrow n_{\text{NG}} = n_{BS}$.

Counting rule for spacetime symmetries

Why is it more complicated ?

- Non-compact symmetries $\rightsquigarrow$ less "nice" mathematical properties.
- The VEV is spacetime dependent $\rightsquigarrow$ functional aspect of QFT is emphasized (minimization of energy etc.).
- Effective Lagrangians less constrained $\rightsquigarrow$ more complicated dispersion relation.
- Derrick's theorem suggests the need of higher derivative terms in the fundamental theory $\rightsquigarrow$ even toy models are complicated!

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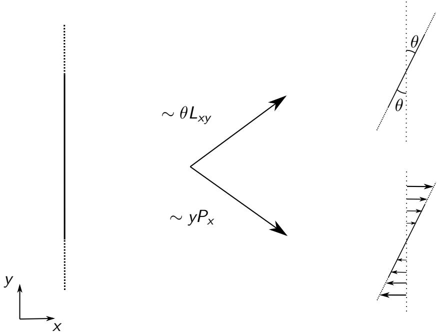
But does it affect the counting rules ?

- Yes! Ex: breaking of the conformal group to Poincaré group $\rightsquigarrow$ 1 NG mode for $d+1$ broken generators.

Counting rule for spacetime symmetries

Some clues for the reduction of NG modes:

Non-dynamical reduction due to definitions and trsf. laws.



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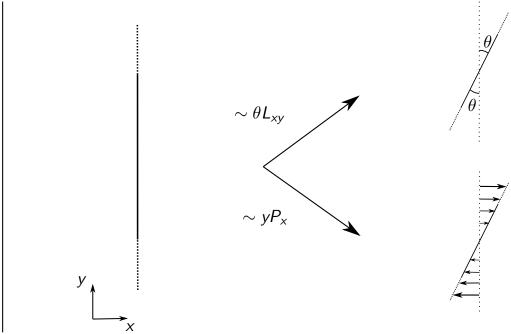


Formalized through the Inverse Higgs Constraints (IHC)

$$\text{If } \boxed{[P_\mu, X_1] \sim X_2} \Rightarrow \text{possible to write } \mathcal{L}_{\text{eff}}(\pi_1) \text{ w/o } \pi_2 \ ,$$

where P_μ an unbroken translation gen. and X_i are broken gen.

Under which criteria impose IHC ?? [Ivanov et al., Teo. Mat. Fiz. '75]

Summary

What to bear in mind:

- SSB $\Rightarrow$ NG candidates π_a $\Rightarrow$ independent massless NG modes.
- π_a transform "shift like".
- $\mathcal{L}_{\text{int}}(\pi) \sim \ldots \partial\pi \ldots$.
- Counting rule for internal SSB but not for spacetime SSB.
- **Possibly** NG reduction through $[P_\mu, X_1] \sim X_2$ (IHC).
- Translation SSB $\rightsquigarrow$ higher derivative terms.

Structure of the talk

Introduction to fractons

Vocabulary:

- Fractonic modes = excitations with reduced mobility.
- Fractons = immobile modes.
- Subdimensional modes = modes which can propagate in a restricted number of directions.

Historically:

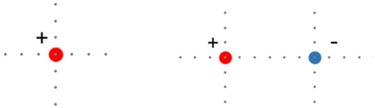
- First noticed in lattice models. [Vijay et al., PRB. '15]
- Then extended to continuous model via (spatial) higher rank gauge theories (A_{ij}). [Pretko, PRB. '17]

Intuition:

- Gauge theories $\Rightarrow$ gauge constraints in the Hamiltonian formalism (ex: Gauss law).
- Gauge constr. interpreted as conservation laws for external sources.
- Higher rank gauge theo. $\Rightarrow$ more cons. laws $\Rightarrow$ restricted dynamics.

$$\partial_i \partial_j E^{ij} = \rho \Rightarrow \begin{cases} \int \rho = \int \partial_i \partial_j E^{ij} = 0 \\ \int x^k \rho = \int x^k \partial_i \partial_j E^{ij} = -\int \partial_j E^{kj} = 0 \end{cases},$$

$\Rightarrow$ Global charge &
dipole moment conservation



Global field theories for fractons

Fractonic global field theories ?

- A signature for fractons: conservation of higher momentum.
- $\exists$ classification of the symmetries leading to it. [Gromov, PRX. '19]
- Simplest case: $\phi(x) \to \phi(x) + \alpha + \beta_i x^i$.
  The Noether charges:
  $$Q \equiv \int \rho \ , \qquad Q^i \equiv \int x^i \rho \ , \qquad \text{with } \rho = j^0_{(\alpha)} \ .$$
- Invariant theories: shift sym. with high derivative terms $\longrightarrow$ NG modes from space translation SSB excellent candidates!

Which leads us to...

Structure of the talk

What do we want to do ?

- SSB of spacetime symmetries $\longrightarrow$ complicated $\longrightarrow$ toy models.
- Concomitant breaking of translation and dilatation.
    - Condensed matter ($\leadsto$ crystals).
    - Stat. physics (around critical points).
- Simplest case: homogeneous breaking $\longrightarrow$ need internal $U(1)$ SSB (+ permits a μ).
- Are there fractons somewhere ?

The model

- In $2+1$ d, Φ a global-$U(1)$ complex scalar field & Ξ a real scalar field.

$$\mathcal{L} = \partial_t\Phi^*\partial_t\Phi + A\partial_i\Phi^*\partial_i\Phi + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi - B\frac{(\partial_i\Phi^*\partial_i\Phi)^2}{\Xi^6} - H\Xi^6 \ ,$$

with A, $B > 0$, and $H = \frac{A^2}{4B}$ $\rightsquigarrow$ permits the energy minimisation.

- W/ natural units, notice that

$$[\Phi] = \frac{1}{2} \ , \qquad [\Xi] = \frac{1}{2} \ , \qquad [A] = [B] = [H] = 0 \ .$$

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- Spatially-dependent stable background & homogeneous SSB $\rightarrow$ 2 kinds:

$$\left(\Phi_0 = \rho \, e^{i k_i x_i} \ , \Xi_0 = v \right) \text{ or } \left(\Phi_0 = b_i x_i \ , \Xi_0 = v \right) \ .$$

- Approx. same qualitative results $\rightarrow$ we discuss here

$$\Phi_0(t, x, y) = \rho \, e^{ikx} \ ,$$
$$\Xi_0(t, x, y) = v \ .$$

Spectrum prediction

$$\mathcal{L} = \partial_t\Phi^*\partial_t\Phi + A\partial_i\Phi^*\partial_i\Phi + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi - B\frac{(\partial_i\Phi^*\partial_i\Phi)^2}{\Xi^6} - H\Xi^6 \ ,$$

- $\left(\Phi_0 = \rho\, e^{ikx} \ , \ \Xi_0 = v\right)$ is a solution if $\frac{k^2\rho^2}{v^6} = \frac{A}{2B}$.

  $\rightsquigarrow$ 2 flat directions $\rightsquigarrow$ we expect 2 NG modes.

- Broken symmetries: $P_1, \ R, \ D, \ S_R, \ S_I, \ Q$.

- But: $P_1 - k\,Q$ is an unbroken translation sym. $\rightsquigarrow n_{\text{BS}} = 5$.

- Considering the IHCs:

$$[P_1 - kQ, D] \propto P_1, \qquad [P_2, R] \propto P_1,$$
$$[P_1 - kQ, S_R] \propto S_I, \qquad [P_1 - kQ, S_I] \propto S_R \ .$$

$\rightsquigarrow n_{\text{BS}}^{(\text{in})} = 2$.

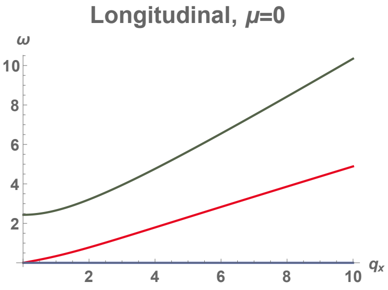
Perturbative computation

- Small fluctuations around the background:

$$\Phi(t,x,y) = \rho\, e^{ikx}\left[1+\sigma(t,x,y)+i\chi(t,x,y)\right] \;,$$
$$\Xi(t,x,y) = v\,\left[1+\tau(t,x,y)\right] \;.$$

- Abusive nomenclature: χ the $U(1)$-mode, σ the shifton & τ the dilaton. Consistent w/ Ward Ids. (cf. later).

- Numerics:



$A = 0.125 \;,\; B = 0.25 \;,\; k = 1.5 \;,\; \rho = 1.07$

Ward-Takahashi Identities

- 3 Ward. Ids. $\sim$ 3 EOMs but when $q \gg k$, un-mixing $\rightsquigarrow$ useful for modes identification.

- $U(1)$ Ward Id. at linear order:

$$\partial_t^2\chi - 2A\,\partial_x^2\chi \simeq 0\ ,\ \text{with}\ k \to 0\ ,$$
$$\Rightarrow \omega^2 \simeq 2Aq_x^2\ , \qquad q \gg k\ .$$

- Real shift Ward Id. at linear order:

$$\partial_t^2\sigma \simeq 0\ ,\ \text{with}\ k \to 0\ ,$$
$$\Rightarrow \omega^2 \simeq 0\ , \qquad q \gg k\ .$$

- Dilatation Ward Id. at linear order (based on improved $T_{\mu\nu}$):

$$\partial_i\partial^i\tau - \partial_t^2\tau \simeq 0\ ,\ \text{with}\ k \to 0\ ,$$
$$\Rightarrow \omega^2 \simeq q_x^2 + q_y^2\ , \qquad q \gg k\ .$$

Connection to fractons

- Quadratic action in the fluctuations:

$$\mathcal{L}_{\text{quad}} = \frac{v^2}{2}\partial_\mu\tau\partial^\mu\tau + \rho^2(\partial_t\chi)^2 + \rho^2(\partial_t\sigma)^2 - 2A\rho^2\left[\partial_x\chi + k(\sigma - 3\tau)\right]^2 .$$

- Emergent coordinate-dependent shift symmetries:

$$\delta\chi = \alpha(y) + \beta(x,y), \;\; \delta\sigma = -\frac{1}{k}\partial_x\beta(x,y) + 3\delta + 3\gamma_i x^i, \;\; \delta\tau(x,y) = \delta + \gamma_i x^i \; ,$$

where $\beta(x, y)$ & $\alpha(y)$ are arbitrary functions.

- $\beta(x, y) \to$ imobile mode (fracton)
  $\alpha(y) \to$ 1-dimensional submode (lineon).

- At higher order, we lose this enhanced sym. $\to$ fractonic modes need interactions to propagate.

Connection to fractons

$$\mathcal{L}_{\text{quad}} = \frac{v^2}{2}\partial_\mu\tau\partial^\mu\tau + \rho^2(\partial_t\chi)^2 + \rho^2(\partial_t\sigma)^2 - 2A\rho^2\left[\partial_x\chi + k(\sigma - 3\tau)\right]^2 .$$

<u>Toward the effective theory</u>:

- Diagonalization of the mass terms:

$$\begin{pmatrix} v\tau \\ \sqrt{2}\,\rho\sigma \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \eta \\ \varphi \end{pmatrix} , \text{ with } \tan\theta = \frac{v}{3\sqrt{2}\,\rho} . \quad (*)$$

- η is the massive mode with $m_\eta^2 = 2Ak^2\left(1 + 18\frac{\rho^2}{v^2}\right)$.
- From $(*)$, $v \gg \rho \Rightarrow \sigma \sim \eta$ and $v \ll \rho \Rightarrow \tau \sim \eta$.

Connection to fractons

- By integrating out the massive mode η (low energy expansion):

$$\mathcal{L}_{\text{quad, eff}} = \rho^2(\partial_t\chi)^2 + \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}\sin^2\theta(\partial_i\varphi)^2 + \frac{2\sqrt{A}\rho}{m_\eta}\sin\theta\cos\theta\partial_x\chi\partial_i^2\varphi$$
$$+ \frac{2A\rho^2}{m_\eta^2}\left[(\partial_t\partial_x\chi)^2 - \cos^2\theta(\partial_i\partial_x\chi)^2\right] + \frac{\sin^2\theta\cos^2\theta}{2m_\eta^2}(\partial_i^2\varphi)^2 \ .$$

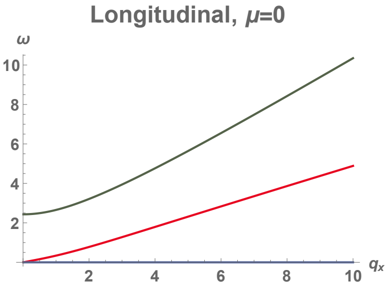
- One of the symmetries:

$$\chi \to \chi + e + a_i x^i + c_{ij}x^i x^j + f(y) \ .$$

- Constant shift $\to$ conserved charge.
  Polynomial shift $\to$ higher moment charge conservation.
  $\rightsquigarrow$ signature of χ as the imobile mode (at low energy).

$$\omega_\chi^2 \simeq 0 \ , \quad \omega_\varphi^2 \simeq \frac{v^2}{18\rho^2 + v^2}q_i^2 \ .$$

Back to the numerics



$A = 0.125$, $B = 0.250$, $k = 1.50$, $\rho = 1.07$

Complete dispersion relations

From the quadratic Lagrangian itself:

Compared to the W.Ids. and the EFT, we get

- ω_3 at low energy.
- Subtlety at high momenta ($q \gg m_\eta$):

$$\omega_2^2 \simeq \begin{cases} q_x^2 + q_y^2 & \text{if} \quad (2A-1)q_x^2 - q_y^2 > 0 \\ 2Aq_x^2 & \text{if} \quad (2A-1)q_x^2 - q_y^2 < 0 \end{cases} ,$$

and

$$\omega_3^2 \simeq \begin{cases} 2Aq_x^2 & \text{if} \quad (2A-1)q_x^2 - q_y^2 > 0 \\ q_x^2 + q_y^2 & \text{if} \quad (2A-1)q_x^2 - q_y^2 < 0 \end{cases} .$$

If $A > \frac{1}{2}$, ω_2 and ω_3 swap their behaviour according to the direction in the (q_x, q_y) plane.

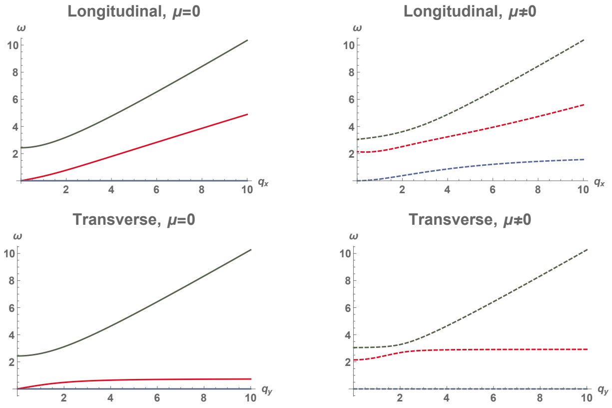
$\Rightarrow$ The identification of the NG candidates with the modes diagonalizing the kinetic matrix depends on the norm **and** the direction of $\vec{q}$.

A word about $\mu \neq 0$ case

- Small fluctuations around the background:

$$\Phi = \rho\, e^{i(\mu t + kx)} \left[1 + \sigma + i\chi\right] \;, \Xi = v\, \left[1 + \tau\right] \;.$$

- Numerics: $A = 0.125 \;, \; B = 0.25 \;, \; k = 1.5 \;, \; \mu = 1 \;, \; \lambda = 0.5$



- Analytically: $\boxed{m_2^2 \sim \mu^{8/3} k^{-2/3} \;.}$

Structure of the talk

Conclusion

Fracton perspective:

- NG modes transforms "shift like".
- SSB of translation requires higher derivatives.
- Associated EFTs are good candidates to display polynomial shift $\rightsquigarrow$ fractonic signature!
- Indeed, from a non-fractonic toy model, fractons emerged!

Goldstone perspective:

- Inverse Higgs Constrains have to be considered.
- Highly non-trivial modes identification.
- We have a gap scaling with μ but "ugly" $\neq$ internal case.

Outlooks

- Generalization of the toy model
  (has already been partially done in the paper).
- Exotic cases $\rightsquigarrow$ holography.
- Phenomenological models similar to our toy model ?

Thank you!

Switching on a chemical potential

Why ?

- Conceptually interesting (comparison w/ internal sym.).
- To get closer to standard lab. conditions.

How to handle a μ:

- At thermal equilibrium:

$$Z = \mathrm{Tr}\left[e^{-\beta(H-\mu Q)}\right] \ .$$

where μ is the chem. pot. associated to $U(1)$.

- At $T = 0$ ($\beta \to +\infty$), the thermal vac. satisfies:

$$(H - \mu Q)\,|\mu\rangle = 0 \ .$$

- If $U(1)$ SSB, then $|\mu\rangle \sim e^{i\mu t}$.
- $\Rightarrow$ in $\mathcal{L}$, term $\sim \mu^2 \Rightarrow$ add. term to stabilise.

Spectrum prediction

$$\mathcal{L} = \partial_t\Phi^*\partial_t\Phi + A\partial_i\Phi^*\partial_i\Phi + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi - B\frac{(\partial_i\Phi^*\partial_i\Phi)^2}{\Xi^6} - H\Xi^6 - \lambda^2(\Phi^*\Phi)^3 \,,$$

- $\left(\Phi_0 = \rho\, e^{i(\mu t + kx)}\,,\ \Xi_0 = v\right)$ is a solution if $\frac{k^2\rho^2}{v^6} = \frac{A}{2B}$ & $\mu^2 = 3\rho^4\lambda^2$.

  $\rightsquigarrow$ 1 flat direction $\rightsquigarrow$ we expect 1 NG modes.

- Broken symmetries: $P_0,\ P_1,\ R,\ D,\ Q$.

- But: $P_1 - k\,Q$ & $P_0 - \mu\,Q$ are unbroken translation sym. $\rightsquigarrow n_{\mathrm{BS}} = 3$.

- Considering the IHCs:

$$[P_1 - kQ, D] \propto P_1, \qquad [P_2, R] \propto P_1,$$

$\rightsquigarrow n_{\mathrm{BS}}^{(\mathrm{in})} = 1$.

Perturbative computation

- Small fluctuations around the background:

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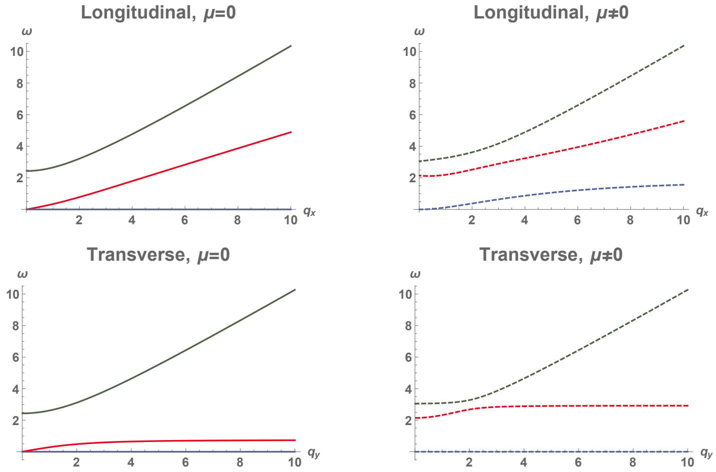
- Numerics: $A = 0.125$, $B = 0.25$, $k = 1.5$, $\mu = 1$, $\lambda = 0.5$



Spectrum discussion

- Rem: analytics highly involved $\to$ only parts of the spectrum.

Some highlights:

- Enhanced shift sym. at quadratic order is lost.
- For $\omega, k, \mu \ll q_x$, $\omega_1^2 \approx 4\mu^2$.
- For $\mu \ll k$, $m_3^2 = m_\eta^2$, $\boxed{m_2^2 \sim \mu^{8/3} k^{-2/3}}$
  $\rightsquigarrow$ "ugly" scale w/. μ, $\neq$ internal case.
- Smooth limit $\mu \to 0$, $\lambda \to 0$ with $\mu/\lambda = cst.$ reproduces $\mu = 0$ case.
  $\rightsquigarrow$ when $\mu \ll k$ and/or $\mu \ll q$, the results are qualit. the same as $\mu = 0$.
- $\mu \gg k \gg q$ has been dealt w/. $\rightsquigarrow$ confirm a hierarchy between m_3 & m_2.